Inequalities: Mixing Variables

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1. Introduction to Mixing Variables (MV)

As a warm-up exercise, find the equality case for each of the following inequalities. (Don't worry about proving the inequalities for now. We will be doing that soon.)

1. Let a, b, c be non-negative real numbers not all zero. Prove that
$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab} \geq \frac{8}{(a+b+c)^2}.$$

2. Let $x, y, z \geq 0$ such that $xy+yz+zx=1$. Prove that
$$\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2} \geq \frac{9}{4}.$$

3. Let x, y, z be pairwise distinct non-negative real numbers. Prove that
$$\frac{1}{(x-y)^2}+\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2} \geq \frac{4}{xy+yz+zx}.$$

Determining most or all equality cases for an inequality is an important first step in solving most inequalities. To see why, we see that $a=b=c$ is not an equality condition for the first inequality. Hence, the following step cannot be taken to solve the inequality.

$$\sum_{cyc}\frac{1}{2a^2+bc} \geq \sum_{cyc}\frac{1}{2a^2+\frac{b^2+c^2}{2}} = \sum_{cyc}\frac{2}{4a^2+b^2+c^2}.$$

In the inequality we just performed, equality holds if and only if $a=b=c$. But that is not an equality case for the inequality we are trying to solve.[1] Hence, we just performed an inequality that resulted in a term is less than the right-hand side $\frac{8}{(a+b+c)^2}$, when $a=b\neq 0$, $c=0$.

Oops. We better undo and reconsider a better step to take.

Important Lesson in Solving Inequalities
The conditions for which equality holds dictates which inequalities you can use to solve the original inequality.

[1] We can make the same argument even if $a=b=c$ is an equality case but other equality cases exist.

Before continuing, we will review the equality cases for certain well known inequalities.

QM-AM-GM-HM Inequality:
Let $x_1, \cdots, x_n$ be positive real numbers. Then

$$\sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}} \geq \frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n} \geq \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}.$$

Equality holds if and only if $x_1 = \cdots = x_n$.

Cauchy-Schwarz Inequality:
Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be any real numbers. Then

$$(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) \geq (a_1 b_1 + \cdots + a_n b_n)^2.$$

Equality holds if and only if $a_i = kb_i$ for some constant k, for each $i \in \{1, \cdots, n\}$.

Muirhead's Inequality:
Let x, y, z be non-negative real numbers. Let a, b, c, a', b', c' be non-negative real numbers such that $a \geq b \geq c, a' \geq b' \geq c', a + b + c = a' + b' + c', a \geq a', a + b \geq a' + b'$. Then

$$\sum_{sym} x^a y^b z^c \geq \sum_{sym} x^{a'} y^{b'} z^{c'}.$$

The following are the equality cases:

1. If $(a, b, c) = (a', b', c')$, then clearly equality always holds.
2. Else if $c, c' > 0$, then equality holds if and only if $x = y = z$ or at least one of x, y, z is 0.
3. Else if $c = c' = 0$ and $b > 0$, then equality holds if and only if $x = y = z$ or if two of x, y, z are equal and the third is equal to 0.
4. Otherwise, equality holds if and only if $x = y = z$.

Schur's Inequality:
Let x, y, z be non-negative real numbers. Then for any real number r, we have

$$x^r(x - y)(x - z) + y^r(y - z)(y - x) + z^r(z - x)(z - y) \geq 0.$$

If $r \geq 0$, equality holds if and only if $x = y = z$ or $(x, y, z) = (t, t, 0)$ for any fixed $t \geq 0$ or its permutations. If $r < 0$, equality holds if and only if $x = y = z$.

The following is the list of common equality conditions for inequalities.

1. All three variables are equal.
2. Exactly two of the variables are equal.
3. At least one variable is the boundary of the domain of the variables. For example, if $x, y, z \geq 0$, then at least one of x, y, z is equal to 0 in the equality case.

Let's revisit a very simple inequality.

Problem 1: Let $x, y, z \geq 0$. Prove that $x + y + z \geq 3\sqrt[3]{xyz}$.

Solution: Let $f(x, y, z) = x+y+z-3\sqrt[3]{xyz}$. Without loss of generality, suppose $z = \min\{x, y, z\}$.

Let $t = \frac{x+y}{2}$. Note that $t \geq z$. We will prove that $f(x, y, z) \geq f(t, t, z) \geq 0$. Then

$$f(x, y, z) - f(t, t, z) = (x + y + z - t - t - z) - 3\sqrt[3]{xyz} + 3\sqrt[3]{t^2 z} = 3\sqrt[3]{z}(\sqrt[3]{t^2} - \sqrt[3]{xy}).$$

Since $t^2 = (\frac{x+y}{2})^2 = \frac{(x+y)^2}{4} \geq \frac{4xy}{4} = xy$, this term is non-negative. Hence, $f(x, y, z) \geq f(t, t, z)$.

We have that $f(t, t, z) = 2t + z - 3\sqrt[3]{t^2 z}$. Let $u = \sqrt[3]{t}, v = \sqrt[3]{z}$. Since $t \geq z$, $u \geq v$. Therefore, $f(t, t, z) = 2u^3 - 3u^2 v + v^3 = (u - v)(2u^2 + 2uv - v^2) \geq 0$. $\square$

Similarly, we can solve Problem 1 using the same method, but using $t = \sqrt{xy}$ instead.

Let's observe what we just did in solving Problem 1

1. We rewrote the inequality in the form $f(x, y, z) \geq c$, for some constant c.
2. We made assumptions on x, y, z based on the symmetric property of the inequality. For example, setting $z = \min\{x, y, z\}$.
3. We then chose an appropriate value of t and prove that $f(x, y, z) \geq f(t, t, z) \geq c$, i.e. we reduce the inequality down to two variables.

Using this technique, let's dissect an earlier inequality problem.

Problem 2: Let a, b, c be non-negative real numbers which are not all zero. Prove that

$$\frac{1}{2a^2 + bc} + \frac{1}{2b^2 + ca} + \frac{1}{2c^2 + ab} \geq \frac{8}{(a + b + c)^2}.$$

Solution: Let

$$f(x, y, z) = \frac{1}{2x^2 + yz} + \frac{1}{2y^2 + zx} + \frac{1}{2z^2 + xy} - \frac{8}{(x + y + z)^2}.$$

Recall that we conjectured that our equality case occurs at $(a, b, c) = (t, t, 0)$ and its permutations. Let's assume $a \geq b \geq c$. Let $t = \frac{a+b}{2}$. [2]

As an exercise, you will now prove the following:

1. $f(a, b, c) \geq f(t, t, c)$.
2. $f(t, t, c) \geq 0$ for all t, c, not both zero.

This of course solves Problem 2. Note that $t \geq c$ and $a, b \geq c$ when proving the inequality. You will be using these properties to solve the problem in this manner! $\square$

We will present an example where we use the same technique to solve an inequality that has initial conditions on a, b, c. We learn how to be clever in the algebraic manipulation when dealing with the initial conditions.

Problem 3: Let $a, b, c \geq 0$ such that $ab + bc + ca = 1$. Prove that

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} \geq \frac{9}{4}.$$

Without loss of generality, suppose $a \geq b \geq c$. Note that $(a, b, c) = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ and $(1, 1, 0)$ are both equality cases that satisfy the condition $ab + bc + ca = 1$. So we need to choose t such that $f(a, b, c) \geq f(t, t, c) \geq 9/4$.

But how do we choose t? Note that (t, t, c) must satisfy the initial condition $ab + bc + ca = 1$. Since $a = b$ in both equality cases, we want to choose t such that if $a = b = t$, then $ab + bc + ca = 1$ is satisfied. Hence, we choose t such that $t^2 + 2tc = 1$. Note that $a \geq t \geq b \geq c$. (Why?)

We can certainly express t in terms of c to prove that $f(a, b, c) \geq f(t, t, c)$. Unfortunately, this results in t being expressed as the square root of another expression. The expansion can become quite ugly. However, we can be clever and create some identities to make the algebraic manipulations easier.

- $t^2 + 2ct = 1 = ab + bc + ca \Leftrightarrow (t + c)^2 = (a + c)(b + c)$

Then

$$f(a, b, c) - f(t, t, c) = \frac{1}{(a+b)^2} - \frac{1}{4t^2} + \frac{1}{(b+c)^2} + \frac{1}{(a+c)^2} - \frac{2}{(a+c)(b+c)}$$

$$= \frac{4t^2 - (a+b)^2}{4(a+b)^2 t^2} + \frac{(a-b)^2}{(a+c)^2(b+c)^2} = \frac{(2t-a-b)(2t+a+b)}{4(a+b)^2 t^2} + \frac{(a-b)^2}{(a+c)^2(b+c)^2}$$

[2] Why this choice of t, as oppose to say, $t = \sqrt{xy}$? It is hard to precisely explain this choice. But you do want to choose t such that expanding $f(a, b, c) - f(t, t, c)$ is "nice". If your choice of t results in extremely ugly expansions, you may want to reconsider another choice for t.

Looks like we are stuck. Or are we? Let's do something clever again.

- $a+b-2t = (a+c)+(b+c)-2(c+t) = (a+c)+(b+c)-2\sqrt{(a+c)(b+c)} = (\sqrt{a+c}-\sqrt{b+c})^2 = (a-b)^2/(\sqrt{a+c}+\sqrt{b+c})^2$.

Why do we do this? Well, we know that $a = b$ holds for all conjectured equality cases to $f(a,b,c) \geq f(t,t,c)$. It should not be surprising that we want the term $(a-b)^2$ to appear. Let's continue expanding $f(a,b,c) - f(t,t,c)$.

$$f(a,b,c) - f(t,t,c) = (a-b)^2 \left(\frac{1}{(a+c)^2(b+c)^2} - \frac{2t+a+b}{4(\sqrt{a+c}+\sqrt{b+c})^2(a+b)^2t^2} \right). \qquad (1)$$

But since $c \leq t$ and $b \leq t \leq a$, we have

$$(\sqrt{a+c}+\sqrt{b+c})^2 = (a+c)+(b+c)+2\sqrt{(a+c)(b+c)} = (a+c)(b+c)+2(t+c) \geq a+b+2t$$

and

$$(a+c)^2(b+c)^2 = (t+c)^2(t+c)^2 \leq (2t)^2(a+b)^2 = 4t^2(a+b)^2.$$

Hence, substituting these two inequalities into (1) yields $f(a,b,c) - f(t,t,c) \geq 0$.

We will leave as an exercise for the reader to prove that $f(t,t,c) \geq \frac{9}{4}$. $\square$

2. Mixing Variables in Boundaries Conditions

We will now handle the case where inequalities have their equality cases hold when at least one of the variables is on the boundary of their range. The simplest example is when $x, y, z \geq 0$, then equality holds when one of the terms is 0. We will begin with a problem from the Canadian Mathematical Olympiad 1999.

Problem 4: Let $x, y, z \geq 0$ such that $x+y+z = 1$. Prove that

$$x^2y + y^2z + z^2x \leq \frac{4}{27}.$$

The first thing we do of course is to find the equality case for this inequality. Clearly, $x = y = z$ does not work. A little bit of playing around would yield that $(x,y,z) = (2/3, 1/3, 0)$ and its cyclic solutions are the equality cases. At this point we conjecture that these are the only equality cases.

We first assume $x \geq y, z$. We need to deal with one nuance to cyclic, non-symmetric inequalities. Let $f(x,y,z) = x^2y + y^2z + z^2x$. First, prove that if $y < z$, then $f(x,y,z) \leq f(x,z,y)$. Hence, we may assume that $y \geq z$.

A nice technique is to push the solution to its boundaries as follows; let $f(x,y,z) = x^2y + y^2z + z^2x$. Then we want to find x', y' such that

$$f(x,y,z) \le f(x', y', 0)$$

for some suitable x', y' and $(x', y', 0)$ satisfying the initial conditions of the problem, if there are any. In this case, x', y' satisfy $x' + y' = 1$. Now how do we choose x', y' ? The following are all reasonable possibilities.

1. Prove $f(x,y,z) \le f(x+z, y, 0)$?
2. Prove $f(x,y,z) \le f(x, y+z, 0)$?
3. Prove $f(x,y,z) \le f(x + \frac{z}{2}, y + \frac{z}{2}, 0)$?

Solution 1: By a little work, we see that $f(x+z, y, 0) - f(x,y,z) = xz(2y - z) - yz(y - z) \ge 0$, since $x \ge y \ge z$. Hence, we may assume that $z = 0$. Now proceed to finish the problem.

Solution 2: By a little work, we see that $f(x + \frac{z}{2}, y + \frac{z}{2}, 0) - f(x,y,z) = \frac{z^3}{8} + \frac{z^2y}{4} + \frac{xz}{2}(x - z) + yz(x - y) \ge 0$. Again, we may assume that $z = 0$. $\square$

Try the following problem.

Problem 5: Let x, y, z be pairwise distinct non-negative real numbers. Prove that

$$\frac{1}{(x-y)^2} + \frac{1}{(y-z)^2} + \frac{1}{(z-x)^2} \ge \frac{4}{xy + yz + zx}.$$

Let $f(x,y,z) = LHS - RHS$. Find conditions on x, y, z (by using symmetry) such that $f(x,y,z) - f(x,y,0) \ge 0$. Hence, we find ourselves a way to make one variable zero. Try it.

3. Entirely Mixing Variables (EMV)

Let's revisit Problem 5 to demonstrate how to entirely mix variables (EMV).

An interesting observation is that the left-hand side of the inequality in Problem 5 is dependent only on the pairwise difference between x, y and z and not on the actual values of x, y, z. Hence, if we replace x, y, z with $x - d, y - d, z - d$, respectively, for any fixed real number d, then the left-hand side remains the same. What about the right-hand side? Note that decreasing x, y, z increases the right-hand side, but keeps the left-hand side the same. How does this help us solve the inequality?

Without loss of generality, suppose $x > y > z$. If we decrease x, y, z each by z (or equivalently, assume $z = 0$) and prove the resulting inequality, we have solved our problem! Let

$$f(x,y,z) = \frac{1}{(x-y)^2} + \frac{1}{(y-z)^2} + \frac{1}{(z-x)^2} - \frac{4}{xy + yz + zx}.$$

Assume $x > y > z$. Then we are trying to prove that

$$f(x, y, z) \geq f(x - z, y - z, 0) \geq 0.$$

The problem has been reduced to proving

$$\frac{1}{(x-y)^2} + \frac{1}{y^2} + \frac{1}{x^2} \geq \frac{4}{xy}.$$

This is now much easier to solve. Do it!

In general, EMV involves make the terms $x - y, y - z, z - x$ appear as much as possible so that we can increase/decrease x, y, z by any fixed constant and see what happens to the terms not dependent on the pairwise difference of x, y, z. Sometimes you have to do some clever factoring or some algebraic manipulation to make the terms $x - y, y - z, z - x$ appear. See Problem 9.

If you need to prove $f(x, y, z) \geq c$, you may instead need to prove $f(x, y, z) \geq f(x-d, y-d, z-d) \geq c$ for some appropriately chosen d.

Final Words

In summary, Mixing Variables is a technique to reduce an inequality by one less variable. The resulting inequality should be much easier, though not necessarily non-trivial.

Handle with care when mixing variables. It is adviseable to check your work carefully as you are writing up your inequality. Another important note is that MV cannot solve every inequality. Be prepared to abandon this strategy if it is not working.

We only have time at the camp to touch on EMV. There is another powerful technique in mixing variables called Stronger Mixing Variables (SMV). It is a powerful technique to solve inequalities with three variables and higher. See the attached references for notes on these topics.

References

[1] Hung, Pham Kim, *The entirely mixing variables method*,
http://reflections.awesomemath.org/2006_5/2006_5_entirelymixing.pdf

[2] Hung, Pham Kim, *The stronger mixing variables method*,
http://reflections.awesomemath.org/2006_6/2006_6_mixing.pdf

Exercises

1. Let $a, b, c \geq 0$ such that $a + b + c = 1$. Prove that
$$\sqrt{a + (b - c)^2} + \sqrt{b + (c - a)^2} + \sqrt{c + (a - b)^2} \geq \sqrt{3}.$$

2. Let x, y, z be positive real numbers in the range $[1, 2]$. Prove that
$$(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 6\left(\frac{x}{y + z} + \frac{y}{z + x} + \frac{z}{x + y}\right).$$

3. Let $a, b, c \geq 0$ such that $ab + bc + ca = 1$. Prove that
$$\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \geq \frac{5}{2}.$$

4. Let A, B, C be angles of a non-obtuse triangle. Prove that
$$\left(\frac{\sin A \sin B}{\sin C}\right)^2 + \left(\frac{\sin B \sin C}{\sin A}\right)^2 + \left(\frac{\sin C \sin A}{\sin B}\right)^2 \geq \frac{9}{4}$$

5. Let $x, y, z \geq 0$. Prove that
$$(x^2 + y^2 + z^2)^2 \geq 4(x + y + z)(x - y)(y - z)(z - x).$$

6. Let $x, y, z \geq 0$ with $x + y + z = 3$. Prove that
$$(x^3 + y^3 + z^3)(x^3y^3 + y^3z^3 + z^3x^3) \leq 36(xy + yz + zx).$$

7. Let $x, y, z \geq 0$ such that $x + y + z = 3$. Prove that
$$(x^2 - xy + y^2)(y^2 - yz + z^2)(z^2 - zx + x^2) \leq 12.$$

8. Let x, y, z be pairwise distinct non-negative real numbers such that $x + y + z = 6$. Prove that
$$\frac{x + y}{(x - y)^2} + \frac{y + z}{(y - z)^2} + \frac{z + x}{(z - x)^2} \geq \frac{3}{2}.$$

9. Let x, y, z be any pairwise distinct real numbers. Prove that
$$(x^2 + y^2 + z^2)\left(\frac{1}{(x - y)^2} + \frac{1}{(y - z)^2} + \frac{1}{(z - x)^2}\right) \geq \frac{9}{2}.$$

10. Let x, y, z be any pairwise distinct real numbers. Prove that
$$\left(\frac{a - b}{b - c}\right)^2 + \left(\frac{b - c}{c - a}\right)^2 + \left(\frac{c - a}{a - b}\right)^2 > \frac{a + b}{b + c} + \frac{b + c}{c + a} + \frac{c + a}{a + b}.$$